Feb 09, 2017

Thursday, February 9, 2017 9:20 AM

Recall an equivalence of continuity $f: X \longrightarrow Y$

$$\forall E \subset X, \; f(\overline{E}) \subset \overline{f(E)}$$

In the proof of this statement, inevitably, we need two definitions.

Continuity at x_0: $\forall V \in \mathcal{T}_Y$ with $f(x_0) \in V$, we have $U \in \mathcal{T}_X$, $x_0 \in U \subset f^{-1}(V)$.

Closure, $x_0 \in \overline{E}$: $\forall U \in \mathcal{T}_X$ with $x_0 \in U$, $U \cap E \neq \phi$

i.e. we have $e \in U \cap E$

Combining these concepts,

$$f(x_0) \in V \in \mathcal{T}_Y \Longrightarrow x_0 \in U \in \mathcal{T}_X, \; U \subset f^{-1}(V)$$

$$\Downarrow$$

$$f(e) \in V \cap f(E) \Leftarrow e \in U \cap E$$

The above argument indicates "points near x_0 will be send by f to points near $f(x_0)$".

In analysis, there is a familiar statement,

$$f(\lim x) = \lim f(x)$$

Definition. A sequence in X is

$$\begin{aligned} \mathbb{N} &\longrightarrow X \\ n &\longmapsto x_n \end{aligned}$$

denoted by $(x_n)_{n=1}^{\infty}$ or $\{x_n\}_{n=1}^{\infty}$

02-09-p2

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Definition. A sequence **converges to** $x \in X$; or x is a **limit**; denoted $x_n \to x$; $\lim_{n\to\infty} x_n = x$ if $\forall$ $U \in \mathcal{T}$ with $x \in U$, $\exists\, N \in \mathbb{N}$, $\forall\, n \geq N$ $x_n \in U$

↳ can be (i) nbhd N of x
or (ii) $U \in \mathcal{U}_x$, local base at x.

First Theorem about limit.

Limit is unique if the space is Hausdorff.

Idea of proof.

Assume $x_n \to x$ and $x_n \to y$, $x \neq y$

By Hausdorff, get $U_1, U_2 \in \mathcal{T}$

$x \in U_1$, $y \in U_2$, $U_1 \cap U_2 = \emptyset$

Argue that for sufficiently large n,

$x_n \in U_1$ and $x_n \in U_2$ Contradiction

Proposition. If $\exists$ $x_n \in A$ and $x_n \to x$ then $x \in \overline{A}$

Idea. Any $U \in \mathcal{T}$ with $x \in U$ contains $x_n \in A$ for large n, $\therefore U \cap A \neq \emptyset$

Corollary. Any convergent sequence in a closed set has its limit in the set.

Question. Is the converse to the proposition also true?

02-09-p3

Monday, February 13, 2017 11:26 PM

Bad Example. Consider $(\mathbb{R}, \text{co-countable})$

(i) $0 \in \overline{\mathbb{R}\setminus\{0\}}$ because the only closed sets are $\mathbb{R}, \phi$, countable sets

$\therefore \overline{\mathbb{R}\setminus\{0\}} = \mathbb{R}$

Alternatively, let $0 \in U$ where $U \in \mathcal{T}$.

Then $\mathbb{R}\setminus U$ is countable, $\therefore U$ is uncountable

$U \cap (\mathbb{R}\setminus\{0\}) = U\setminus\{0\}$ is also uncountable

(ii) $x_n \in \mathbb{R}\setminus\{0\}$ with $x_n \longrightarrow 0$ leads to contradiction

For $U \in \mathcal{T}$ with $0 \in U$, $\exists N \in \mathbb{N}$, $\forall n \geq N$, $x_n \in U$

Then, from above, let $W = U \setminus \{x_n : n \geq N\}$

$\mathbb{R}\setminus W = (\mathbb{R}\setminus U) \cup \{x_n : n \geq N\}\setminus\{0\}$ is countable

$\therefore W \in \mathcal{T}$ and $0 \in W$, but only $x_1, \cdots, x_{N-1} \in W$.

Proposition. Let $f: X \longrightarrow Y$ be continuous

If $x_n \in X$, $x_n \longrightarrow x$ then $f(x_n) \longrightarrow f(x)$

Idea. Take $V \in \mathcal{T}_Y$ and $f(x) \in V$, then

$f^{-1}(V) \in \mathcal{T}_X$ and $x \in f^{-1}(V)$

$\therefore$ Tail of $x_n \in f^{-1}(V)$.

Thus, Tail of $f(x_n) \in V$.

Remark. Converse is not true. That is, even whenever $x_n \to x$ always implies $f(x_n) \longrightarrow f(x)$

f may not be continuous.